On the Well-Foundedness of Adequate Orders

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Adequate Orders

Aim: finite complete prefixes of unfoldings

Definition
A strict partial order $\prec$ on the finite configurations of the unfolding of a Petri net is called adequate if:

- it refines (strict) set inclusion $\subset$,
  $C \subset C' \implies C \prec C'$;

- it is preserved by finite extensions,
  \[
  \begin{cases}
  C \prec C' \\
  \text{Mark}(C) = \text{Mark}(C') \\
  E \sim E'
  \end{cases} \implies C \oplus E \prec C' \oplus E';
  \]
Adequate Orders

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- it is preserved by finite extensions,
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  \end{align*} \implies C \oplus E \prec C' \oplus E'; \]

- it is well founded.
The Case of Safe Petri Nets

**Theorem**

Well-foundedness of $\triangle$ is a consequence of the other requirements.
A Corollary of Preservation by Finite Extensions

Definition
For a linearisation $u$ of a configuration $C$, denote $\sigma(u) \in (\mathcal{RM} \times T)^*$ the word:

$((\text{current marking}, \text{next transition}), \ldots, (\text{current marking}, \text{next transition}))$

Definition
$C \leftarrow C'$ if there are linearisations $u$ and $u'$ such that $\sigma(u)$ is a strict subword of $\sigma(u')$.

(subword: erase letters, like $\text{BREAD}$)

Theorem
In safe Petri nets, $C \leftarrow C' \implies C \triangleright C'$
A Corollary of Preservation by Finite Extensions

Proof

\[ u = e_1 \ldots e_{|u|} \]
\[ u' = e'_1 \ldots e'_{|u'|} \]

There exist \( 1 \leq i_1 < \cdots < i_{|u|+1} = |u'| + 1 \) s.t. \( \sigma(u)_n = \sigma(u')_{i_n} \)

Denote \( C_n \overset{\text{def}}{=} \{e_1, \ldots, e_n\} \)
\[ C'_n \overset{\text{def}}{=} \{e'_1, \ldots, e'_{i_{n+1}-1}\} \]

Let \( j \) be the smallest index such that \( i_j \neq j \).

\[ \begin{align*}
\triangleright & \quad C_{j-1} \subset C'_{j-1}, \text{ then } C_{j-1} \triangleleft C'_{j-1} \\
\triangleright & \quad \{e_j\} \sim \{e'_i\} \text{ and } \text{Mark}(C_{j-1}) = \text{Mark}(C'_{j-1}), \text{ then } \\
& \quad C_{j-1} \oplus \{e_j\} \succeq C'_{j-1} \oplus \{e'_i\} \\
\triangleright & \quad \begin{cases}
C_{j-1} \oplus \{e_j\} = C_j \\
C'_{j-1} \oplus \{e'_i\} \subseteq C' \\
\end{cases}, \text{ then } C_j \triangleleft C' \\
\triangleright & \quad \ldots \\
\triangleright & \quad C_n \triangleleft C'_n, \text{ i.e. } C \triangleleft C' 
\end{align*} \]
The Case of Safe Petri Nets

Proof

1. Assume $C_1 \triangleright C_2 \triangleright \ldots$
2. There exist $i < j$ such that $C_i \leftarrow C_j$
3. $C_i \triangleleft C_j$: contradiction

Detail of point 2

- Assume $|C_1| < |C_2| < \ldots$
- For each $n$, let $u_n$ be a linearisation of the events of $C_n$.
- By Higman’s lemma, there exist $i, j$ such that $\sigma(u_i)$ is a subword of $\sigma(u_j)$.

Higman’s lemma
In any infinite set of finite words over a finite alphabet, there exist two words $u$ and $v$ such that $u$ is a subword of $v$. 
Weak vs. strong preservation by finite extensions

**Strong preservation:**
\[
\forall C \triangleleft C' \text{ such that } \text{Mark}(C) = \text{Mark}(C') \\
\forall E' \forall E \sim E' \quad C \oplus E \triangleleft C' \oplus E'
\]

**Weak preservation:**
\[
\forall C \triangleleft C' \text{ such that } \text{Mark}(C) = \text{Mark}(C') \\
\forall E' \exists E \sim E' \quad C \oplus E \triangleleft C' \oplus E'
\]

Weak preservation ensures completeness
Counter-example with Weak Preservation

\[ p_1 \xrightarrow{a} \]
\[ p_2 \xrightarrow{b} \]

\[ p_1 \xrightarrow{a} p_1 \xrightarrow{b} p_2 \xrightarrow{b'} p_2' \]
\[ \cdots \]

\[ n \quad a^n \quad a^n b \quad a^n b' \]

\[
\begin{array}{c}
\cdots \\
3 \\
2 \\
1 \\
0 \\
n \\
a^n \\
a^n b \\
a^n b'
\end{array}
\]
## Summary of the results

<table>
<thead>
<tr>
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<th>weak preservation</th>
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Counter-example with strong preservation (unbounded net)

\[
p_1 \quad a \quad p_1 \quad a \quad p_1 \quad a \quad p_1 \quad \cdots
\]

\[
b \quad p_2 \quad b \quad p_2 \quad b \quad p_2 \quad \cdots
\]

\[
0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad n \quad a^n \quad a^n b
\]
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The Case of Bounded Petri Nets with Strong Preservation

Theorem
Well-foundedness of \( \triangleleft \) is a consequence of the other requirements.
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Even Stronger Preservation

**Definition**

Extendible order:

\[ \begin{align*}
C & \triangleleft C' \\
E & \sim E'
\end{align*} \implies C \oplus E \triangleleft C' \oplus E' \]

(even if \( \text{Mark}(C) \neq \text{Mark}(C') \))

**Theorem**

*Well-foundedness of \( \triangleleft \) is a consequence of the other requirements.*
## Summary of the results

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Variants

Preservation by single-event extensions

Other isomorphisms: pomset, Parikh

- Sufficient for completeness
- The results are not affected
Conclusion

- Interest: no need to prove well-foundedness
- Works in the most common cases

Remarks

- Single-event extensions are sufficient for completeness.
- Variants of isomorphisms do not affect the results.
- Simpler proofs with pre-order $\sqsubseteq$ instead of strict order $\lhd$. 