On the Well-Foundedness of Adequate Orders

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Adequate Orders

Aim: finite complete prefixes of unfoldings

Definition

A strict partial order $\lhd$ on the finite configurations of the unfolding of a Petri net is called adequate if:

- it refines (strict) set inclusion $\subset$,
  $$C \subset C' \implies C \lhd C';$$
- it is preserved by finite extensions,
  $$\left.\begin{array}{c} C \lhd C' \\ Mark(C) = Mark(C') \\ E \sim E' \end{array}\right\} \implies C \oplus E \lhd C' \oplus E';$$

Adequate Orders

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$$\left\{ \begin{array}{c} C \lhd C' \\ Mark(C) = Mark(C') \\ E \sim E' \end{array} \right\} \implies C \oplus E \lhd C' \oplus E';$$
- it is well founded.

The Case of Safe Petri Nets

Theorem

Well-foundedness of $\lhd$ is a consequence of the other requirements.

A Corollary of Preservation by Finite Extensions

Definition

For a linearisation u of a configuration C, denote $\sigma(u) \in (\mathcal{RM} \times T)^*$ the word: ((current marking, next transition), …, (current marking, next transition))

Definition

$C \leftharpoondown C'$ if there are linearisations u and u' such that $\sigma(u)$ is a strict subword of $\sigma(u')$.

(subword: erase letters, like ~~B~~REA~~A~~D)

Theorem

In safe Petri nets, $C \leftharpoondown C' \implies C \lhd C'$

A Corollary of Preservation by Finite Extensions

Proof

$u = e_1 \ldots e_{|u|}$
$u' = e'_1 \ldots e'_{|u'|}$
There exist $1 \le i_1 < \cdots < i_{|u|+1} = |u'| + 1$ s.t. $\sigma(u)_n = \sigma(u')_{i_n}$

Denote $C_n \overset{\text{def}}{=} \{e_1, \ldots, e_n\}$
$C'_n \overset{\text{def}}{=} \{e'_1, \ldots, e'_{i_{n+1}-1}\}$

Let j be the smallest index such that $i_j \ne j$.

- $C_{j-1} \subset C'_{j-1}$, then $C_{j-1} \lhd C'_{j-1}$
- $\{e_j\} \sim \{e'_{i_j}\}$ and $Mark(C_{j-1}) = Mark(C'_{j-1})$, then $C_{j-1} \oplus \{e_j\} \lhd C'_{j-1} \oplus \{e'_{i_j}\}$
- $\begin{Bmatrix} C_{j-1} \oplus \{e_j\} = C_j \\ C'_{j-1} \oplus \{e'_{i_j}\} \subseteq C'_j \end{Bmatrix}$, then $C_j \lhd C'_j$
- ...
- $C_n \lhd C'_n$, i.e. $C \lhd C'$

The Case of Safe Petri Nets

Proof

1. Assume $C_1 \rhd C_2 \rhd \ldots$
2. There exist $i < j$ such that $C_i \leftharpoonup C_j$
3. $C_i \lhd C_j$: contradiction

Detail of point 2

- Assume $|C_1| < |C_2| < \ldots$
- For each n, let u_n be a linearisation of the events of C_n.
- By Higman's lemma, there exist i, j such that $\sigma(u_i)$ is a subword of $\sigma(u_j)$.

Higman's lemma

In any infinite set of finite words over a finite alphabet, there exist two words u and v such that u is a subword of v.

The Case of Unsafe Petri Nets

Weak vs. strong preservation by finite extensions

Strong preservation:

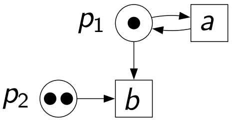
$$\forall C \lhd C' \text{ such that } \mathit{Mark}(C) = \mathit{Mark}(C')$$
$$\forall E' \quad \forall E \sim E' \quad C \oplus E \lhd C' \oplus E'$$

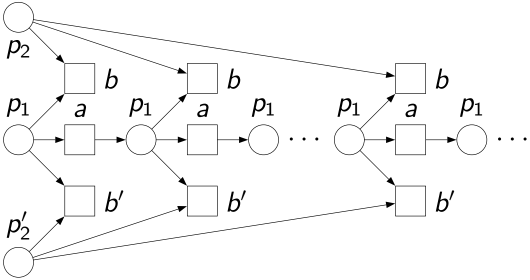
Weak preservation:

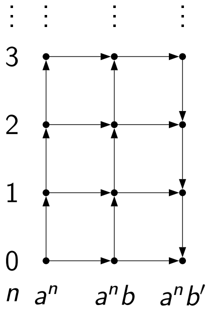
$$\forall C \lhd C' \text{ such that } \mathit{Mark}(C) = \mathit{Mark}(C')$$
$$\forall E' \quad \exists E \sim E' \quad C \oplus E \lhd C' \oplus E'$$

Weak preservation ensures completeness

Counter-example with Weak Preservation



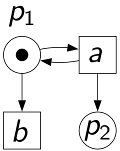


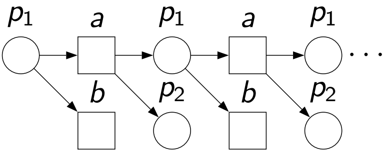


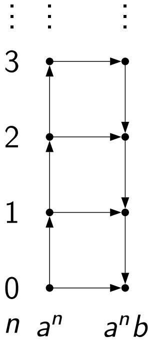
Summary of the results

	weak preservation	strong preservation
safe	✓	
unsafe	×	?

Counter-example with strong preservation (unbounded net)







Summary of the results

	weak preservation	strong preservation
safe	✓	
bounded	×	?
unbounded	×	

The Case of Bounded Petri Nets with Strong Preservation

Theorem

Well-foundedness of $\lhd$ is a consequence of the other requirements.

Summary of the results

	weak preservation	strong preservation
safe	✓	
bounded	×	✓
unbounded	×	

Even Stronger Preservation

Definition

Extendible order:

$$\left\{\begin{matrix} C \lhd C' \\ E \sim E' \end{matrix}\right\} \implies C \oplus E \lhd C' \oplus E'$$

(even if $Mark(C) \neq Mark(C')$)

Theorem

Well-foundedness of $\lhd$ *is a consequence of the other requirements.*

Summary of the results

<table>
<tr><th></th><th>weak preservation</th><th>strong preservation</th><th>extendible order</th></tr>
<tr><td>safe</td><td colspan="3">✓</td></tr>
<tr><td>bounded</td><td>×</td><td colspan="2">✓</td></tr>
<tr><td>unbounded</td><td colspan="2">×</td><td>✓</td></tr>
</table>

Variants

Preservation by single-event extensions

Other isomorphisms: pomset, Parikh

- Sufficient for completeness
- The results are not affected

Conclusion

- Interest: no need to prove well-foundedness
- Works in the most common cases

Remarks

- Single-event extensions are sufficient for completeness.
- Variants of isomorphisms do not affect the results.
- Simpler proofs with pre-order $\trianglelefteq$ instead of strict order $\triangleleft$.